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The effects of combined horizontal and vertical heterogeneity on the onset of convection in a porous medium: Moderate heterogeneity

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Abstract

The effects of hydrodynamic and thermal heterogeneity, for the case of variation in both the horizontal and vertical directions, on the onset of convection in a horizontal layer of a saturated porous medium uniformly heated from below, are now studied analytically for the case of moderate heterogeneity (rather than the weak heterogeneity previously studied), for the case of a square box where the properties vary in a piecewise constant or linear fashion, with conducting impermeable top and bottom boundaries and insulating impermeable sidewalls. In order to allow for the moderate heterogeneity the order of the Galerkin expansion employed has been increased, and the expansion of a determinant of high order has been avoided by the use of a least squares methodology to find the critical value of the Rayleigh number *Ra*. It is found that the effects of permeability heterogeneity and conductivity heterogeneity each cause a reduction in the critical value of *Ra* in all cases, and the effects of horizontal and vertical heterogeneity are still approximately additive.

1. Introduction

Studies of the effects of heterogeneity in the classical Horton–Roger–Lapwood problem (the onset of convection in a horizontal layer of a saturated porous medium uniformly heated from below) have been surveyed by Nield and Bejan [1]. The pioneering study was that of Gheorghitza [2], and particularly notable are the studies of vertical heterogeneity (especially the case of horizontal layers) by McKibbin and O'Sullivan [3,4], McKibbin and Tyvand [5–7], Nield [8] and Leong and Lai [9,10], and the studies of horizontal heterogeneity by McKibbin [11], Nield [12] and Guonot and Caltagirone [13]. For completeness, we mention that some more general aspects of conductivity heterogeneity have been discussed by Vadasz [14], Braester and Vadasz [15] and Rees and Riley [16]. Until recently the interaction between vertical heterogeneity and horizontal heterogeneity had not been investigated, and this is the focus of our current investigation.

The topic of permeability heterogeneity in particular is currently of interest for an additional reason. Simmons et al. [17] and Prasad and Simmons [18] have pointed out that in many heterogeneous geologic systems, hydraulic properties such as the hydraulic conductivity of the system under consideration can vary by many orders of magnitude and sometimes rapidly over small spatial scales. They also pointed out that the onset of instability in transient, sharp interface problems is controlled by very local conditions in the vicinity of the evolving boundary layer and not by the global layer properties or indeed some average property of that macroscopic layer. They also pointed out that any averaging process would remove the very structural controls and physics that are expected to be important in controlling the onset, growth, and/or decay of instabilities in a highly heterogeneous system. In particular, in the case of dense plume migration in highly heterogeneous environments the application of an average global Rayleigh number based upon average hydraulic conductivity of the

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Nomenclature

c	specific heat
k	k^*/k_0
k^*	overall (effective) thermal conductivity
k_0	mean value of $k^*(x^*, y^*)$
K	K^*/K_0
K^*	permeability
K_0	mean value of $K^*(x^*, y^*)$
L	height (and width) of the enclosure
P	dimensionless pressure, $\frac{(\rho c)_f K_0}{\mu k_0} P^*$
P^*	pressure
Ra	Rayleigh number, $\frac{(\rho c)_f \rho_0 g \beta K_0 L (T_1 - T_0)}{\mu k_0}$
t^*	time
t	dimensionless time, $\frac{k_0}{(\rho c)_m L^2} t^*$
T^*	temperature
T_0	temperature at the upper boundary
T_1	temperature at the lower boundary
u	dimensionless horizontal velocity, $\frac{(\rho c)_m L}{k_0} u^*$
$\mathbf{u}^*$	vector of Darcy velocity, (u^*, v^*)
v	dimensionless vertical velocity, $\frac{(\rho c)_m L}{k_0} v^*$
x	dimensionless horizontal coordinate, x^*/L
x^*	horizontal coordinate
y	dimensionless upward vertical coordinate, y^*/L
y^*	upward vertical coordinate

Greek symbols

β	fluid volumetric expansion coefficient
θ	dimensionless temperature, $\frac{T^* - T_0}{T_1 - T_0}$
μ	fluid viscosity
ρ	density
ρ_0	fluid density at temperature T_0
σ	heat capacity ratio, $\frac{(\rho c)_m}{(\rho c)_f}$
ψ	streamfunction

Subscripts

f	fluid
m	overall porous medium

Superscript

*	dimensional variable

medium is problematic. In these cases, an average Rayleigh number is unable to predict the onset of instability accurately because the system is characterized by unsteady flows and large amplitude perturbations.

Nield and Simmons [19] have emphasized the need to distinguish between weak heterogeneity and strong heterogeneity. For the case of weak heterogeneity (properties varying by a factor not greater than 3 or so) the introduction of an equivalent Rayleigh number is useful. The extent to which an equivalent Rayleigh number (based on averaged permeability and averaged conductivity) might work was investigated by Nield [8] for the case of vertical heterogeneity. He concluded that provided the variation of each of the various parameters lies within one order of magnitude, a rough and ready estimate of an effective Rayleigh number can be made that is useful as a criterion for Rayleigh–Bénard convection. This effective Rayleigh number is based on the arithmetic mean of quantities (such as the permeability) that appear in the numerator, and the harmonic mean of quantities (such as the viscosity) that appear in the denominator of the defining expression. Similar conclusions were drawn by Leong and Lai [9,10]. In the case of strong heterogeneity the concept of an effective Rayleigh number loses validity as a criterion for the onset of instability.

For the case of weak heterogeneity some progress has been made, for the case of two-dimensional convection in a rectangular box with impervious thermally insulated side walls. Based on the expectation that for weak heterogeneity the solution would not differ dramatically from the solution for the homogeneous case, one can utilize an extension, to the case of trial functions of both the horizontal and vertical coordinates, of the Galerkin approximate method that has been widely employed (see, for example, Finlayson [20]). The present authors have used this methodology in a series of papers [21–27], starting from a basic problem using the Darcy model and then extending the analysis to the Brinkman model, the double-diffusive situation, the case of heterogeneity of the basic vertical temperature gradient, the case of local thermal non-equilibrium and the case of a bidisperse porous medium, and also allowing for the effect of heterogeneity of anisotropy. The analysis leads to an eigenvalue equation involving a determinant of large order, but in the case of weak heterogeneity the expansion of the determinant can be handled by neglecting terms that are of order higher than the second in the relevant small quantities.

The present paper treats the case of moderate heterogeneity. The parameters expressing the variation from homogeneity are allowed to be of order unity rather than taken to be small compared with unity. The order of the Galerkin approximation employed is increased. The need to expand a determinant of large order is avoided by using a method of least squares to obtain the minimum Rayleigh number.

2. Analysis

Single-phase flow in a saturated porous medium is considered. Asterisks are used to denote dimensional variables. We consider a square box, $0 \leqslant x^* \leqslant L, 0 \leqslant y^* \leqslant L$, where the y^* axis is in the upward vertical direction. The side walls are taken as insulated, and uniform temperatures T_0 and T_1 are imposed at the upper and lower boundaries, respectively.

Within this box the permeability is $K^*(x^*, y^*)$ and the overall (effective) thermal conductivity is $k^*(x^*, y^*)$. The Darcy velocity is denoted by $\mathbf{u}^* = (u^*, v^*)$. The Oberbeck–Boussinesq approximation is invoked and local thermal equilibrium is assumed. The equations representing the conservation of mass, thermal energy, and Darcy's law take the form

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{1}$$

$$(\rho c)_{\mathrm{m}} \frac{\partial T^*}{\partial t^*} + (\rho c)_{\mathrm{f}} \left[u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right] = \frac{\partial}{\partial x^*} \left[k^*(x^*, y^*) \frac{\partial T^*}{\partial x^*} \right] + \frac{\partial}{\partial y^*} \left[k^*(x^*, y^*) \frac{\partial T^*}{\partial y^*} \right] \tag{2}$$

$$u^* = -\frac{K^*(x^*, y^*)}{\mu} \frac{\partial P^*}{\partial x^*}, \quad v^* = \frac{K^*(x^*, y^*)}{\mu} \left[-\frac{\partial P^*}{\partial y^*} - \rho_0 \beta g (T^* - T_0) \right]. \tag{3a, b}$$

Here $(\rho c)_{\mathrm{m}}$ and $(\rho c)_{\mathrm{f}}$ are the heat capacities of the overall porous medium and the fluid, respectively, μ is the fluid viscosity, ρ_0 is the fluid density at temperature T_0, and β is the volumetric expansion coefficient.

We introduce dimensionless variables by defining

$$(x, y) = \frac{1}{L}(x^*, y^*), \quad (u, v) = \frac{(\rho c)_m L}{k_0}(u^*, v^*), \quad t = \frac{k_0}{(\rho c)_{\mathrm{m}} L^2} t^*, \quad \theta = \frac{T^* - T_0}{T_1 - T_0}, \quad P = \frac{(\rho c)_f K_0}{\mu k_0} P^*, \tag{4a, b, c, d, e}$$

where k_0 is the mean value of $k^*(x^*, y^*)$ and K_0 is the mean value of $K^*(x^*, y^*)$.

We also define a Rayleigh number Ra by

$$Ra = \frac{(\rho c)_{\mathrm{f}} \rho_0 g \beta K_0 L (T_1 - T_0)}{\mu k_0} \tag{5}$$

and the heat capacity ratio

$$\sigma = \frac{(\rho c)_{\mathrm{m}}}{(\rho c)_{\mathrm{f}}}. \tag{6}$$

The governing equations then take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7}$$

$$\frac{\partial \theta}{\partial \tau} + \frac{1}{\sigma} \left[u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} \right] = \frac{\partial}{\partial x} \left[k(x, y) \frac{\partial \theta}{\partial x} \right] + \frac{\partial}{\partial y} \left[k(x, y) \frac{\partial \theta}{\partial y} \right], \tag{8}$$

$$u = -K(x, y) \frac{\partial P}{\partial x}, \quad v = K(x, y) \left[-\frac{\partial P}{\partial y} + \sigma Ra \theta \right], \tag{9}$$

where $k(x, y) = k^*(x^*, y^*)/k_0$ and $K(x, y) = K^*(x^*, y^*)/K_0$.

We introduce a streamfunction ψ so that

$$u = \sigma Ra \frac{\partial \psi}{\partial y}, \quad v = -\sigma Ra \frac{\partial \psi}{\partial x}. \tag{10a, b}$$

We also eliminate P. The result is

$$\frac{\partial}{\partial x} \left[\frac{1}{K(x, y)} \frac{\partial \psi}{\partial x} \right] + \frac{\partial}{\partial y} \left[\frac{1}{K(x, y)} \frac{\partial \psi}{\partial x} \right] + \frac{\partial \theta}{\partial y} = 0, \tag{11}$$

$$\frac{\partial \theta}{\partial \tau} + Ra \left[\frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} \right] = \frac{\partial}{\partial x} \left[k(x, y) \frac{\partial \theta}{\partial x} \right] + \frac{\partial}{\partial y} \left[k(x, y) \frac{\partial \theta}{\partial y} \right]. \tag{12}$$

The conduction solution is given by

$$\psi = 0, \quad \theta = 1 - y. \tag{13a, b}$$

The perturbed solution is given by

$$\psi = \varepsilon \psi', \theta = 1 - y + \varepsilon \theta'. \tag{14a, b}$$

To first order in the small constant ε, we get

$$\frac{\partial}{\partial x} \left[\frac{1}{K(x, y)} \frac{\partial \psi'}{\partial x} \right] + \frac{\partial}{\partial y} \left[\frac{1}{K(x, y)} \frac{\partial \psi'}{\partial y} \right] + \frac{\partial \theta'}{\partial x} = 0, \tag{15}$$

$$\frac{\partial \theta}{\partial \tau} + Ra \frac{\partial \psi'}{\partial x} - \frac{\partial}{\partial x} \left[k(x, y) \frac{\partial \theta'}{\partial x} \right] - \frac{\partial}{\partial y} \left[k(x, y) \frac{\partial \theta'}{\partial y} \right] = 0. \tag{16}$$

For the onset of convection we can invoke the "principal of exchange of stabilities" and hence take the time derivative in Eq. (16) to be zero.

The boundary conditions are

$$\psi' = 0 \text{ and } \theta' = 0 \quad \text{on } y = 0, \tag{17a, b}$$

$$\psi' = 0 \text{ and } \theta' = 0 \quad \text{on } y = 1, \tag{18a, b}$$

$$\psi' = 0 \text{ and } \partial \theta'/\partial x = 0 \quad \text{on } x = 0, \tag{19a, b}$$

$$\psi' = 0 \text{ and } \partial \theta'/\partial x = 0 \quad \text{on } x = 1. \tag{20a, b}$$

This set of boundary conditions is satisfied by

$$\psi'_{mn} = \sin m\pi x \sin n\pi y, \quad m, n = 1, 2, 3, \ldots. \tag{21}$$

$$\theta'_{mn} = \cos m\pi x \sin n\pi y, \quad m, n = 1, 2, 3, \ldots. \tag{22}$$

Hence, we can introduce the Fourier expansions

$$\psi' = \sum_{m=1}^{N} \sum_{n=1}^{N} A_{mn} \sin m\pi x \sin n\pi y, \tag{23}$$

$$\theta' = \sum_{m=1}^{N} \sum_{n=1}^{N} B_{mn} \cos m\pi x \sin n\pi y. \tag{24}$$

Substituting into the steady state form of Eqs. (15) and (16) we obtain

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\pi^2(m^2+n^2)A_{mn}\sin m\pi x\sin n\pi y + K(x,y)\sum_{m=1}^{N}\sum_{n=1}^{N}\pi m B_{mn}\sin m\pi x\sin n\pi y + (K_x/K)\sum_{m=1}^{N}\sum_{n=1}^{N}\pi m A_{mn}\cos m\pi x\sin n\pi y + (K_y/K)\sum_{m=1}^{N}\sum_{n=1}^{N}\pi n A_{mn}\sin m\pi x\cos n\pi y = 0, \quad (25)$$

$$Ra\sum_{m=1}^{N}\sum_{n=1}^{N}\pi m A_{mn}\cos m\pi x\sin n\pi y + k(x,y)\sum_{m=1}^{N}\sum_{n=1}^{N}\pi^2(m^2+n^2)B_{mn}\cos m\pi x\sin n\pi y + k_x(x,y)\sum_{m=1}^{N}\sum_{n=1}^{N}\pi m B_{mn}\sin m\pi x\sin n\pi y - k_y(x,y)\sum_{m=1}^{N}\sum_{n=1}^{N}\pi n B_{mn}\cos m\pi x\cos n\pi y = 0. \quad (26)$$

We now employ a Galerkin method. The expressions (21) and (22) are now regarded as trial functions, and the expressions (25) and (26) are the associated residuals. The expression (25) is made orthogonal to each of the expressions (21), and the expression (26) is made orthogonal to each of the expressions (22).

Thus, one has $2N^2$ homogeneous equations in the $2N^2$ unknowns A_{mn}, B_{mn}, where $m = 1,2,\ldots,N$; $n = 1,2,\ldots,N$. The vanishing of the determinant of coefficients gives the eigenvalue equation determining Ra. Rather than expanding the determinant (something that is impractical when N is large) we calculate an approximate value of Ra in the following way. We force a non-trivial solution of the system of $2N^2$ equations by adding as a constraint the additional equation

$$\sum_{m=1}^{N}\sum_{n=1}^{N}(A_{mn}+B_{mn})-\sigma = 0, \quad (30)$$

where σ is a constant scale factor.

We then have more equations than unknowns, so that the system is over-determined, and thus no exact solution of the augmented system exists. However, we can find the best least squares fit to the augmented system and vary Ra to minimize the error, and hence find the desired eigenvalue. The value of σ can be varied to sharpen the minimum.

Specifically, we have the set of equations

$$\mathbf{Mx} = 0, \quad (31)$$

where

$$\mathbf{x}^{\mathrm{T}} = (A_{11},A_{12},\ldots,A_{1N},A_{21},A_{22},\ldots,A_{2N},\ldots,A_{N1},A_{N2},\ldots,A_{NN},B_{11},B_{12},\ldots,B_{1N},B_{21},B_{22},\ldots,B_{2N},\ldots,B_{N1},B_{N2},\ldots,B_{NN}) \quad (32)$$

$$\mathbf{M} = \begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix} \quad (33)$$

Explicitly, for the case where $K_x/K, K_y/K, k_x, k_y$ are constants, the orthogonality properties of the trial functions lead to

$$(\mathbf{M}_{11})_{mn} = (a^2+b^2)\pi^2\delta_{mn} \quad (34)$$
$$(\mathbf{M}_{12})_{mn} = \pi a I_{abcd} \quad (35)$$
$$(\mathbf{M}_{21})_{mn} = Raa\pi\delta_{mn} \quad (36)$$
$$(\mathbf{M}_{22})_{mn} = \pi^2(a^2+b^2)J_{abcd} \quad (37)$$

where

$$a = 1+[(n-1)/N], \quad b = 1+(n-1)\mathrm{mod}N,$$
$$c = 1+[(m-1)N], \quad d = 1+(m-1)\mathrm{mod}N. \quad (38)$$

Here, $[x]$ denotes the integer part of x and δ_{ij} is the Kronecker delta.

We use the notation

$$\langle f(x,y)\rangle = \int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad (39)$$

and define

$$I_{mnpq} = 4\langle K(x,y)\sin m\pi x\sin n\pi y\sin p\pi x\sin q\pi y\rangle, \quad (40)$$
$$J_{mnpq} = 4\langle k(x,y)\cos m\pi x\sin n\pi y\cos p\pi x\sin q\pi y\rangle. \quad (41)$$

We note that $\langle k(x,y)\rangle = 1$ and $\langle K(x,y)\rangle = 1$. Also,

$$4\langle\sin m\pi x\sin n\pi y\sin p\pi x\sin q\pi y\rangle = \begin{cases}1 & \text{if } m=p \text{ and } n=q\\ 0 & \text{otherwise}\end{cases}, \quad (42)$$

$$4\langle\cos m\pi x\sin n\pi y\cos p\pi x\sin q\pi y\rangle = \begin{cases}1 & \text{if } m=p \text{ and } n=q\\ 0 & \text{otherwise}\end{cases}. \quad (43)$$

For example, in the case $N = 2$ one has

$$\mathbf{M} = \begin{bmatrix}
2\pi^2 & 0 & 0 & 0 & \pi I_{1111} & \pi I_{1211} & 2\pi I_{2111} & 2\pi I_{2211}\\
0 & 5\pi^2 & 0 & 0 & \pi I_{1112} & \pi I_{1212} & 2\pi I_{2112} & 2\pi I_{2212}\\
0 & 0 & 5\pi^2 & 0 & \pi I_{1121} & \pi I_{1221} & 2\pi I_{2121} & 2\pi I_{2221}\\
0 & 0 & 0 & 8\pi^2 & \pi I_{1122} & \pi I_{1222} & 2\pi I_{2122} & 2\pi I_{2222}\\
\pi Ra & 0 & 0 & 0 & 2\pi^2 J_{1111} & 5\pi^2 J_{1211} & 5\pi^2 J_{2111} & 8\pi^2 J_{2211}\\
0 & \pi Ra & 0 & 0 & 2\pi^2 J_{1112} & 5\pi^2 J_{1212} & 5\pi^2 J_{2112} & 8\pi^2 J_{2212}\\
0 & 0 & 2\pi Ra & 0 & 2\pi^2 J_{1121} & 5\pi^2 J_{1221} & 5\pi^2 J_{2121} & 8\pi^2 J_{2221}\\
0 & 0 & 0 & 2\pi Ra & 2\pi^2 J_{1122} & 5\pi^2 J_{1222} & 5\pi^2 J_{2122} & 8\pi^2 J_{2222}
\end{bmatrix} \quad (44)$$

To be specific, we sum the squares of the left-hand sides of equations represented by Eq. (31) together with Eq. (30), and adjust the values of the unknowns A_{mn} and B_{mn} to obtain the minimum value of that sum for a fixed value of Ra. Then we vary Ra to minimize this minimum value.

We apply the procedure to a quartered square with piecewise-constant properties. We consider the case

$$
\begin{aligned}
&K(x,y) = 1-\delta_H-\delta_V, k(x,y) = 1-\varepsilon_H-\varepsilon_V,\\
&\quad \text{for } 0<x<1/2,\ \ 0<y<1/2,\\
&K(x,y) = 1+\delta_H-\delta_V, k(x,y) = 1+\varepsilon_H-\varepsilon_V,\\
&\quad \text{for } 1/2<x<1,\ \ 0<y<1/2,\\
&K(x,y) = 1-\delta_H+\delta_V, k(x,y) = 1-\varepsilon_H+\varepsilon_V,\\
&\quad \text{for } 0<x<1/2,\ \ 1/2<y<1,\\
&K(x,y) = 1+\delta_H+\delta_V, k(x,y) = 1+\varepsilon_H+\varepsilon_V,\\
&\quad \text{for } 1/2<x<1,\ \ 1/2<y<1,\\
&K(x,1/2) = 1-\delta_H, k(x,y) = 1-\varepsilon_H,\\
&\quad \text{for } 0<x<1/2,\\
&K(x,1/2) = 1+\delta_H, k(x,y) = 1+\varepsilon_H,\\
&\quad \text{for } 1/2<x<1,\\
&rK(1/2,y) = 1-\delta_V, k(x,y) = 1-\varepsilon_V,\\
&\quad \text{for } 0<y<1/2,\\
&K(1/2,y) = 1+\delta_V, k(x,y) = 1+\varepsilon_V,\\
&\quad \text{for } 1/2<y<1,\\
&K(1/2,1/2) = 1. \qquad (45)
\end{aligned}
$$

3. Results and discussion

We found that, with good approximations for the starting values of the A_{mn} and B_{mn}, we could employ the Mathematica package to locate a minimum value of Ra for values of N as large as 10 (corresponding to the solution of 200 equations in 200 unknowns). Further, for parameter values within a certain domain, we found convergence as we increased N through the values $2, 4, 6, \ldots$. When convergence occurred, we generally obtained accuracy to at least three significant figures when $N = 4$. According, we have generally presented results for that value of N. We reasoned that the extra effort in going to a larger value of N was not warranted. In fact, we found that locating a minimum value of Ra became a tricky process for large N

Table 1
The values of $S = (Ra - Ra_0)/Ra_0$, where Ra_0 is the value for the homogeneous case, for various combinations of the permeability parameters δ_H, δ_V and the conductivity parameters ε_H, ε_V

$\delta_H, \delta_V, \varepsilon_H, \varepsilon_V$	S	S_{wh}	$\delta_H, \delta_V, \varepsilon_H, \varepsilon_V$	S	S_{wh}
0.1, 0, 0, 0	−0.0126	−0.0128	0.4, 0, 0, 0	−0.1314	−0.2050
0, 0.1, 0, 0	−0.0015	−0.0014	0, 0.4, 0, 0	−0.0212	−0.0219
0.1, 0.1, 0, 0	−0.0139	−0.0142	0.4, 0.4, 0, 0	−0.1464	−0.2269
0, 0, 0.1, 0	−0.0053	−0.0050	0, 0, 0.4, 0	−0.1013	−0.0800
0, 0, 0, 0.1	−0.0098	−0.0086	0, 0, 0, 0.4	−0.1286	−0.1370
0, 0, 0.1, 0.1	−0.0152	−0.0136	0, 0, 0.4, 0.4	−0.2437* ($N = 2$)	−0.2170
0.1, 0, 0.1, 0	−0.0020	−0.0018	0.4, 0, 0.4, 0	−0.0288	−0.0288
0, 0.1, 0, 0.1	−0.0040	−0.0031	0, 0.4, 0, 0.4	−0.0684	−0.0493
0.1, 0.1, 0.1, 0.1	−0.0058	−0.0049	0.4, 0.4, 0.4, 0.4	−0.0901	−0.0781
0.2, 0, 0, 0	−0.0448	−0.0512	0.5, 0, 0, 0	−0.1755	−0.3202
0, 0.2, 0, 0	−0.0055	−0.0055	0, 0.5, 0, 0	−0.0324	−0.0343
0.2, 0.2, 0, 0	−0.0496	−0.0567	0.5, 0.5, 0, 0	−0.1953	−0.3545
0, 0, 0.2, 0	−0.0215	−0.0200	0, 0.5, 0, 0	−0.1780	−0.1250
0, 0, 0, 0.2	−0.0390	−0.0342	0, 0, 0, 0.5	−0.2391	−0.2140
0, 0, 0.2, 0.2	−0.0648	−0.0542	0, 0, 0.5, 0.5	−0.3938* ($N = 2$)	−0.3390
0.2, 0, 0.2, 0	−0.0073	−0.0072	0.5, 0, 0.5, 0	−0.0448	−0.0450
0, 0.2, 0, 0.2	−0.0162	−0.0123	0, 0.5, 0, 0.5	−0.1112	−0.0770
0.2, 0.2, 0.2, 0.2	−0.0230	−0.0195	0.5, 0.5, 0.5, 0.5	−0.1770* ($N = 8$)	−0.1220
0.3, 0, 0, 0	−0.0868	−0.1153			
0, 0.3, 0, 0	−0.0121	−0.0123			
0.3, 0.3, 0, 0	−0.0965	−0.1276			
0, 0, 0.3, 0	−0.0516	−0.0450			
0, 0, 0, 0.3	−0.0871	−0.0770			
0, 0, 0.3, 0.3	−0.1717	−0.1220			
0.3, 0, 0.3, 0	−0.0164	−0.0162			
0, 0.3, 0, 0.3	−0.0372	−0.0277			
0.3, 0.3, 0.3, 0.3	−0.0511	−0.0439			

The values are obtained with the value $N = 4$ except where otherwise specified. The values of S_{wh} (the weak heterogeneity approximation) are calculated from Eq. (47).

because accumulated round-off error could lead to the splitting of a single genuine local minimum into two spurious local minima. We found that a convenient choice for σ was the value $\sigma = 100$ (something of the same order-of-magnitude as Ra).

We have presented our results in terms of the parameter S defined by

$$Ra = Ra_0(1 + S), \tag{46}$$

where $Ra_0 = 4\pi^2$ is the critical value of Ra for the homogeneous case.

For the case where the heterogeneity parameters are small compared with unity, it is known (see the erratum to [21]) that

$$S = -[1.281(\delta_H - 0.625\varepsilon_H)^2 + 0.137(\delta_V - 2.5\varepsilon_V)^2]. \tag{47}$$

This now serves as a "weak heterogeneity approximation".

Values calculated from this expression are included in Table 1 for comparison with those computed by the least squares method. In this table we have presented values for five levels of heterogeneity, 0.1, 0.2, 0.3, 0.4, 0.5. Larger values are not realistic because they correspond to a negative permeability or conductivity in one quadrant of the square. The values indicated by an asterisk distinguish cases where convergence has broken down and to fill the gap in the table we have presented a "best guess" obtained using an alternative value of N. It appears that $|S| < 0.2$ gives an estimate for the domain of convergence.

At each level of heterogeneity, we have presented results for nine combinations of values of the heterogeneity parameters δ_H, δ_V, ε_H, ε_V. These have been grouped into three sets of three combinations. In the first set the permeability parameters δ_H, δ_V have been varied, in the second set the conductivity parameters ε_H, ε_V have been varied, and in the third both permeability and conductivity parameters have been varied.

An obvious feature of the results is that all of the values of S presented are negative. That means that the effect of heterogeneity of any kind is to reduce the value of the critical Rayleigh number. In other words, the effect of heterogeneity for a square box is always destabilizing. (We already knew this for the case of weak heterogeneity since the expression for S in Eq. (47) is negative definite.)

The results in the table show that, as one would expect from Eq. (47), there is little interaction between the horizontal heterogeneities and the vertical heterogeneities. Further, the effects of the combination of vertical and horizontal heterogeneities are approximately additive. The effects of permeability heterogeneities are of the same order of magnitude as those of conductivity heterogeneities. Also as one would expect, the expression in Eq. (47) is a good approximation at the 0.1 level of heterogeneity, and is less good for higher levels, when it generally over-predicts variations resulting from permeability heterogeneities and under-predicts those resulting from conductivity heterogeneities.

4. Conclusions

In [21], we investigated the effect of weak heterogeneity. We have now extended the investigation to the case of moderate heterogeneity. We have found that the weak heterogeneity expression given in Eq. (47) is a qualitatively good approximation for the moderate heterogeneity situation. The case of strong heterogeneity remains as a challenge for a future investigation. (A preliminary step has been made by Nield and Simmons [19], who proposed a practical approximate criterion for instability.)

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